NAME: _____

CLASS: 12MTA_____ or 12MTX_____

CHERRYBROOK TECHNOLOGY HIGH SCHOOL

2020

YEAR 12

AP4

MATHEMATICS ADVANCED

Time allowed – 3 hours plus 10 minutes reading time

General Instructions

- Attempt all questions
- Write your name on the question paper
- Write using black pen
- Calculators approved by NESA may be used
- The NESA reference sheet has been provided
- For questions in Section II, show relevant mathematical reasoning and/or calculations
- Marks may not be awarded for careless, badly arranged, or poorly written work

Total marks: 100

Section I – 10 marks (pages 3 – 5)

- Attempt Questions 1-10
- Allow about 15 minutes for this section

Section II – 90 marks (pages 6 – 29)

- Attempt Questions 11-36
- Allow about 2 hours and 45 minutes for this section

BLANK PAGE

Section I

10 Marks
Attempt Questions 1 to 10
Allow about 15 minutes for this section

Use the multiple choice answer sheet for Questions 1 to 10

1 For what values of x is the curve $f(x) = 2x^3 + x^2$ concave down?

(A) $x < -\frac{1}{6}$
(B) $x > -\frac{1}{6}$
(C) $x < -6$
(D) $x > 6$

2 What is the domain of the function $f(x) = \sqrt{x} + \frac{1}{\sqrt{2-x}}$?

(A) (0, 2)
(B) [0, 2)
(C) (0,2]
(D) [0, 2]

3 Two events, A and B are independent, and $P(A)$ and $P(B)$ are both non-zero.

Which of the following expressions gives the probability that event A occurs and then event B also occurs?

(A) $P(B|A)$
(B) $P(A|B)$
(C) $P(A \cup B)$
(D) $P(A)P(B)$

4 The variance of a discrete random variable X is $\mathrm{Var}(X) = 2.6$

Calculate $\mathrm{Var}(3X - 1)$

(A) 6.8
(B) 7.8
(C) 23.4
(D) 22.4

5 What is $f'(x)$ if $f(x) = 3x^4(4-x)^3$?

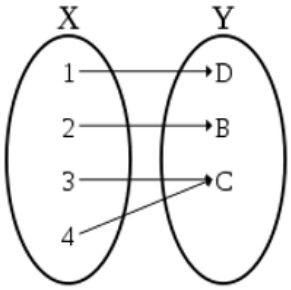
(A) $12x^3(4-x)^2(7x-16)$

(B) $12x^3(4-x)^2(16-7x)$

(C) $3x^3(4-x)^2(7x-16)$

(D) $3x^3(4-x)^2(16-7x)$

6 Which type of relation is shown below?



(A) One-to-one

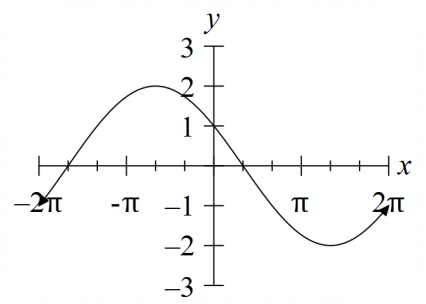
(B) One-to-many

(C) Many-to-one

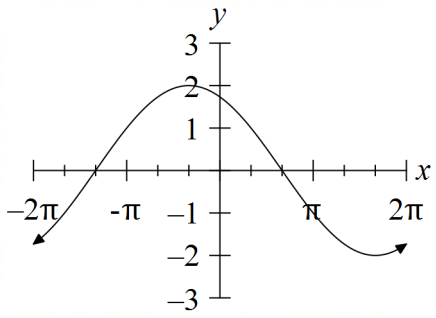
(D) Many-to-many

7 Which of the following best represents the graph of $g(x) = 2\cos\left(\frac{x}{2} + \frac{\pi}{3}\right)$?

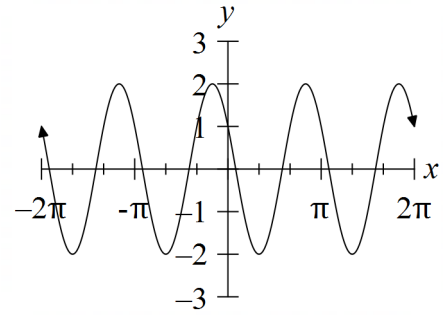
(A)



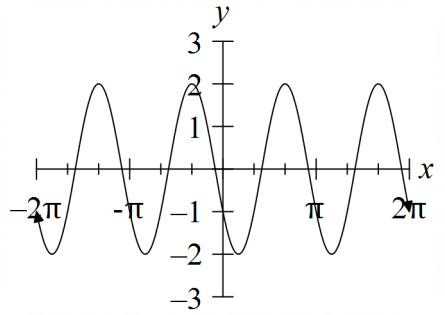
(B)



(C)



(D)



8 What is the value of $\ln 2 + \ln 4 + \ln 8 + \ldots + \ln 2^{2n}$?

(A) $n^2 \ln 2$
(B) $n(n+1)\ln 2$
(C) $n(n+2)\ln 2$
(D) $n(2n+1)\ln 2$

9 What is the equation of the function $f(x)$ whose graph has undergone transformations in the following order?

- Translated left 2 units
- Horizontally dilated by a factor of 3
- Translated down 4 units

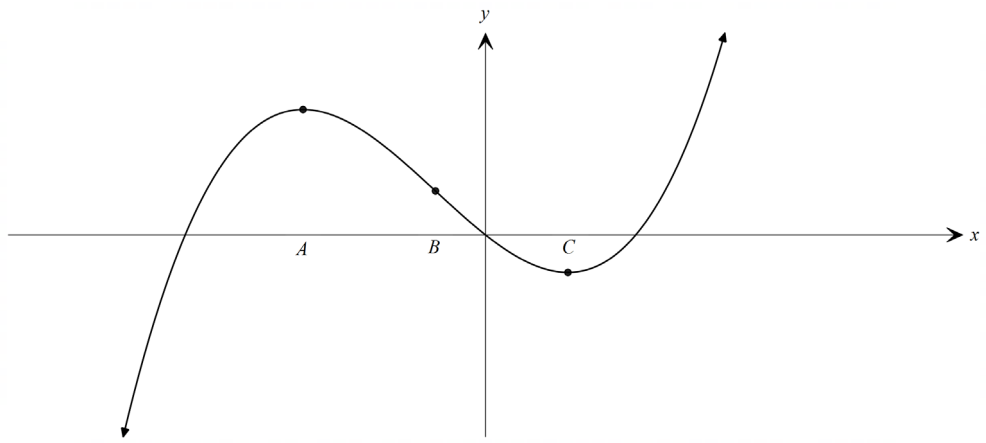
(A) $f\left(\frac{x}{3}+2\right)-4$

(B) $f\big(3(x+2)\big)-4$

(C) $f\left(\frac{x+2}{3}\right)-4$

(D) $f(3x+2)-4$

10 The graph of $y = f(x)$ is shown below.
$x = A$ and $x = C$ are stationary points, and $x = B$ is a point of inflection.



Over what domain is $f'(x) < 0$ and $f''(x) > 0$?

(A) $(-\infty, A)$
(B) (A, B)
(C) (B, C)
(D) (C, ∞)

END OF SECTION I

Section II

90 marks
Attempt Questions 11-36
Allow about 2 hours and 45 minutes for this section

Answer each question in the spaces provided.
Your responses should include relevant mathematical reasoning and/or calculations.
Extra writing space is provided at the back of the examination paper.

Marks

Question 11 (2 marks)

A geometric sequence has a first term of $\frac{1}{8}$ and the 9th term of the sequence is 8192. What is the common ratio? **2**

……

……

……

……

……

……

……

Question 12 (2 marks)

Find the exact value of **2**

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \cos x \, dx.$$

……

……

……

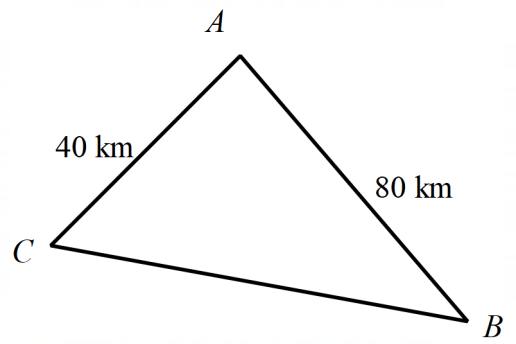
……

……

……

Question 13 (3 marks) **Marks**

Three towns, A, B and C form a triangle. **3**
Town A is 80 km from Town B and Town C is 40 km from Town A as shown below:



The bearing of Town B from Town A is $130°$. The bearing of Town C from Town A is $240°$.

Find the distance between Town B and Town C, to the nearest kilometre.

……

……

……

……

……

……

……

……

……

……

……

……

……

……

……

……

Question 14 (2 marks) **Marks**

Find $\frac{d}{dx}(x^3 \tan 2x)$ **2**

……

……

……

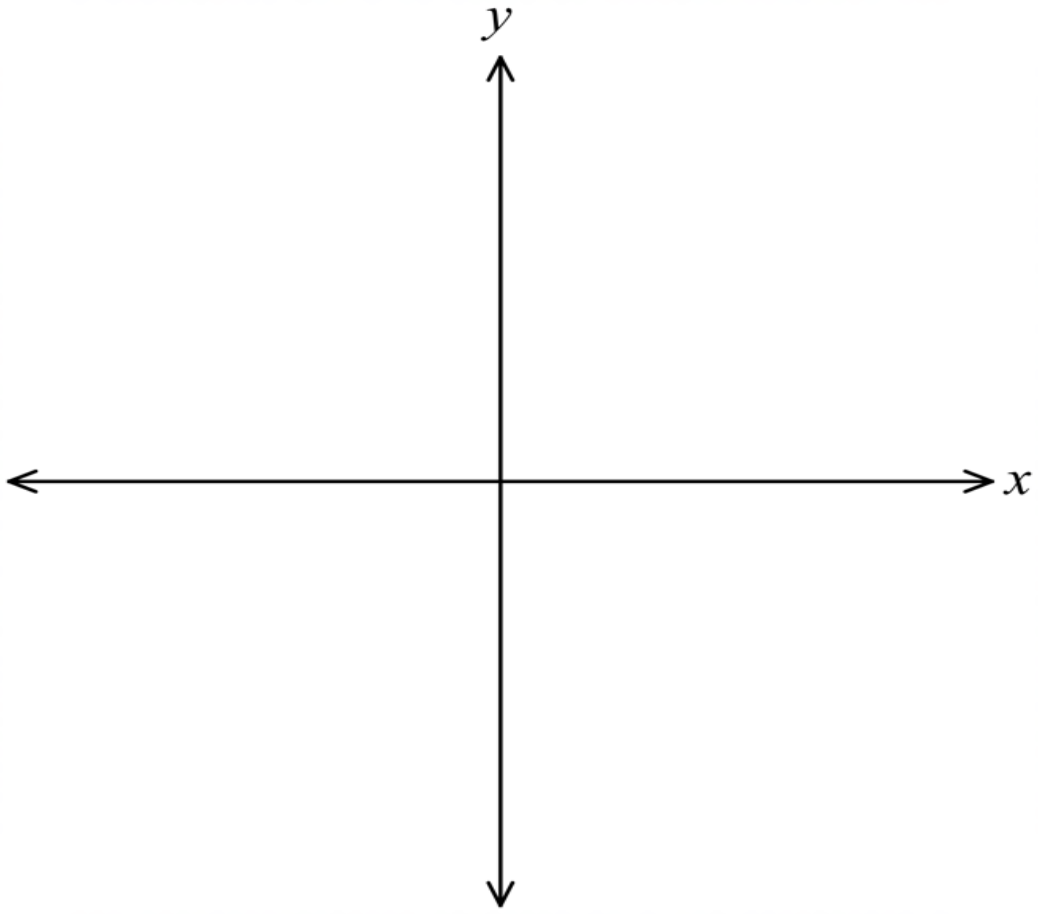
……

……

……

Question 15 (2 marks)

Sketch the graph of $y = -(x+1)(x-2)^3$, showing the intercepts. **2**



Question 16 (2 marks) **Marks**

Given $\cos\theta = \frac{2}{7}$ and $270° \leq \theta \leq 360°$, find the exact value of $\sin\theta$. **2**

……

……

……

……

……

……

……

Question 17 (2 marks)

Show that $(\tan x + \sec x)^2 = \frac{1+\sin x}{1-\sin x}$ **2**

……

……

……

……

……

……

……

……

……

……

……

……

Question 18 (2 marks)

2

Solve $2\cos^2\alpha - 3\cos\alpha - 2 = 0$ for $0 \leq \alpha \leq \pi$

……

……

……

……

……

……

……

……

……

……

Question 19 (2 marks)

Consider the functions $f(x) = e^x$ and $g(x) = \ln(x-2)$

a) Find the composite function $f\big(g(x)\big)$. **1**

……

……

……

……

b) Find in interval notation the range of the composite function. **1**

……

……

……

……

Question 20 (4 marks) **Marks**

A curve with the equation $y = f(x)$, has $\frac{dy}{dx} = x^3 + 2x - 7$.

a) The point P (2, 4) lies on the curve. Find the equation of y. **2**

……

……

……

……

……

……

……

……

b) Find the equation of the normal to the curve at point P, giving your answer in general form. **2**

……

……

……

……

……

……

……

……

……

……

……

……

Question 21 (7 marks) **Marks**

A function is given by $y = 2x^3 + 3x^2 - 12x - 5$

a) Find the stationary points, determine their nature, and find any points of inflection. **5**

Question 21 continues on page 13

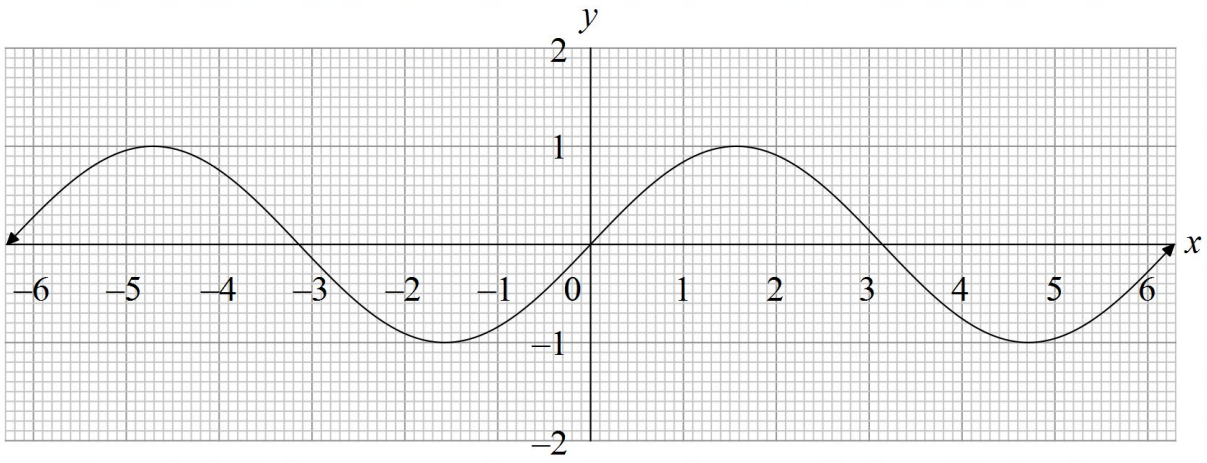
Question 21 Continued **Marks**

b) Sketch the curve, showing the stationary points, the point(s) of inflection and the y-intercept. **2**

End of Question 21

Question 22 (2 marks) **Marks**

The graph of $y = \sin x$ is shown in the diagram.



a) Explain why the equation $\sin x = 1 - \frac{x}{4}$ has three solutions. **1**

……

……

……

b) Use the graph to approximate the largest solution to the equation $\sin x = 1 - \frac{x}{4}$. **1**

……

……

Question 23 (2 marks)

If $y = \frac{e^{3x}}{x+1}$ find $\frac{dy}{dx}$ **2**

……

……

……

……

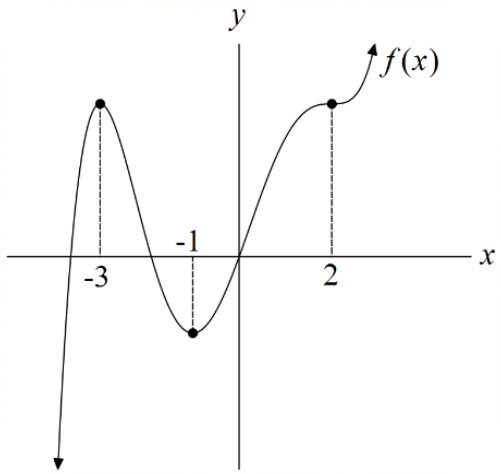
……

……

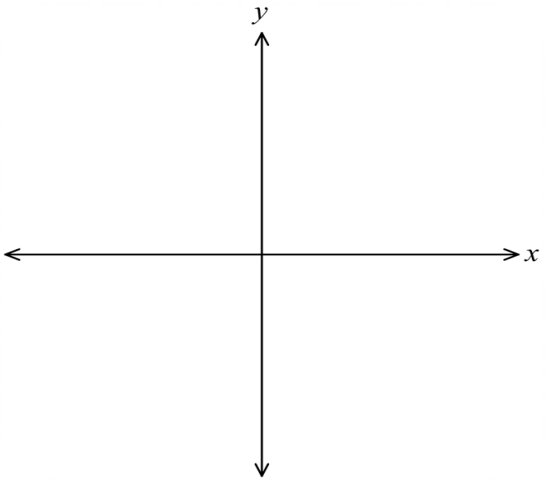
……

Question 24 (4 marks) **Marks**

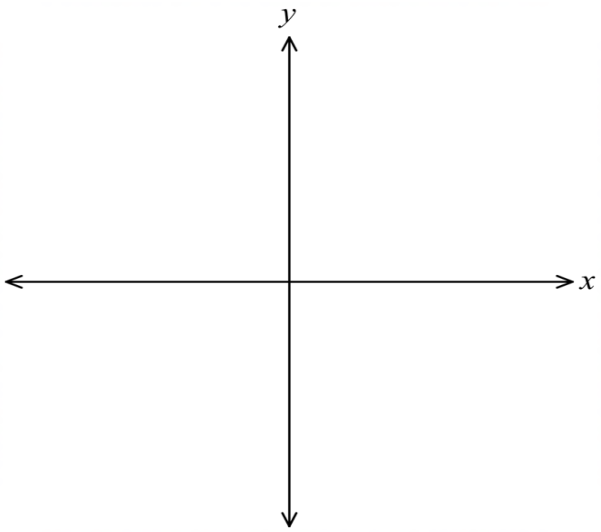
The diagram below shows the graph of $\boldsymbol{y = f(x)}$.



a) Sketch $f'(x)$ **2**



b) Sketch $f(-x)$ **2**



Question 25 (5 marks) **Marks**

a) Show that the derivative of $\ln\left(\frac{3+x}{3-x}\right)$ is $\frac{6}{9-x^2}$ **3**

……

……

……

……

……

……

……

……

……

……

……

……

……

……

……

b) Hence or otherwise find $\int \frac{1}{9-x^2}\,dx$. **2**

……

……

……

……

……

……

Questions 11- 25 are worth 43 marks in total.

Question 26 (3 marks) **Marks**

Use the Trapezoidal rule, with four sub-intervals, to estimate the value of $\int_0^2 \sqrt{4-x^2}\,dx$, correct to 3 decimal places. **3**

……

……

……

……

……

……

……

……

Question 27 (2 marks)

The probability density function for the continuous random variable X is given by:

$$f(x) = \begin{cases} |1-x| & 0 \le x \le 2 \\ 0 & \text{otherwise} \end{cases}$$

Find $P(X \le 1.5)$ **2**

……

……

……

……

……

……

……

……

Question 28 (3 marks) **Marks**

A soft drink company produces a bottled drink. The amount of liquid poured into each bottle by the bottling machine is normally distributed, with a mean of 300 mL and a standard deviation of 10 mL.

a) A bottle has a z-score of -1.2. How many mL below the mean is this bottle? **1**

……

……

……

……

……

b) In order to comply with local regulations, at least 97.5% of bottles must contain more liquid than indicated on the label. **2**

What is the largest amount of millilitres the bottle's label should show in order to comply with the regulations?

……

……

……

……

……

……

……

……

……

……

……

……

……

Question 29 (5 marks) **Marks**

It is known at the beginning of winter in a large population, 15% of the people in the population will be infected with a particular virus.

a) Two people are selected at random, find the probability that both of them have the virus. **1**

……

……

b) Four people are selected at random, find the probability that at least one of them has the virus. Give your answer to 3 decimal places. **2**

……

……

……

……

……

c) What is the smallest number of people a drug company would need to test to have a greater than 95% chance that at least one of the tested people had the virus? **2**

……

……

……

……

……

……

……

……

……

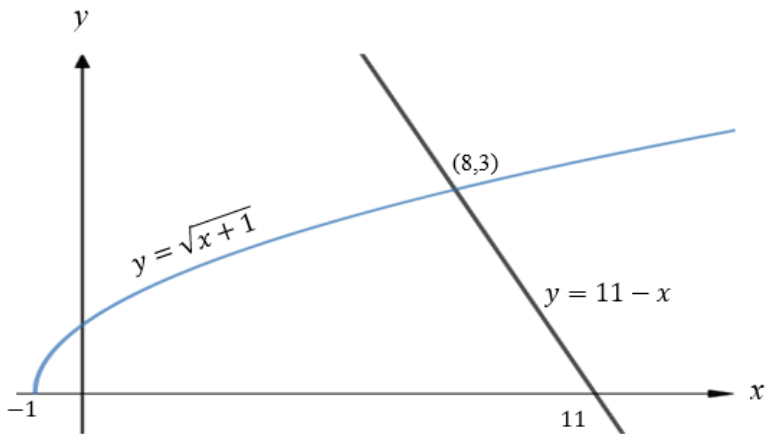
……

……

……

Question 30 (3 marks) **Marks**

The functions $y = \sqrt{x+1}$ and $y = 11 - x$ are sketched below.



Calculate the area bounded by the curves $y = \sqrt{x+1}$ and $y = 11 - x$ and the x-axis. **3**

……

……

……

……

……

……

……

……

……

……

……

……

……

……

……

……

Question 31 (2 marks) **Marks**

Find the value(s) of m given that m, $3m$ and $m^2 + 20$ are consecutive terms of a geometric sequence. **2**

……

……

……

……

……

……

……

Question 32 (5 marks)

A probability density function is defined as $f(x) = ke^{-x}$ in the domain $[0,3]$.

a) Show that $k = 1.0524$, correct to four decimal places. **2**

……

……

……

……

……

……

……

……

……

……

Question 32 continues on page 22

Question 32 Continued **Marks**

b) Hence, find the median value of the probability density function, correct to three decimal places. **3**

……

……

……

……

……

……

……

……

……

……

……

……

……

……

……

……

……

……

……

……

……

……

……

End of Question 32

Question 33 (7 marks) **Marks**

A six-sided die is biased as shown in the probability distribution below.

x	1	2	3	4	5	6
$P(X = x)$	0.1	0.25	0.05	a	0.17	0.13

a) Explain why $a = 0.3$. **1**

……

……

……

……

b) Find $P(2 < X \leq 4)$ **1**

……

……

……

……

c) Find $P(X \leq 4 | X > 2)$ **1**

……

……

d) Find the expected value of X. **1**

……

……

……

……

……

Question 33 continues on page 24

Question 33 Continued **Marks**

e) Find $E(4X + 1)$ **1**

……

……

……

……

……

……

……

f) Find the variance of X. **2**

……

……

……

……

……

……

……

……

……

……

……

……

……

End of Question 33

Question 34 (5 marks) **Marks**

Max conducted a survey of a group of people he knew about their age and how much they earn each week. The results are shown in the table below.

Age (years) (x)	18	45	28	15	32	68
Wage (\$/week) (W)	715	2350	1530	438	1690	1320

a) Using your calculator, find (r) Pearson's correlation coefficient correct to 2 decimal places and describe the type and strength of correlation this data gives. **2**

……

……

……

b) Using your calculator, find the equation of the least-squares regression line in the form **1**

$W = Bx + A$ where A and B are integers.

……

……

c) Use your equation to estimate the earnings of a 50 year-old worker. **1**

……

……

d) Your equation from part (b) cannot be used to make valid estimates for ages greater than 68 and less than 15 years. **1**

Justify this statement with calculations and/or reasons.

……

……

……

……

……

……

Question 35 (6 marks) **Marks**

A swimming pool is to be emptied for maintenance. The quantity of water, Q in litres, remaining in the pool at a time, t minutes, is given by:

$$Q(t) = 2000(25 - t)^2, \quad 0 \leq t \leq 25$$

a) At what rate (in litres/min) is the water being removed at any time (t)? **1**

……

……

……

b) How long will it take to remove at least half of the water from the pool? **2**
Answer to the nearest minute.

……

……

……

……

……

……

……

……

……

……

……

……

Question 35 Continues on page 27

Question 35 Continued **Marks**

c) At what time does the rate of flow of water from the pool reach 20 kL/minute? **2**

……

……

……

.

……

……

……

……

d) Describe how the amount of water remaining in the pool changes as the pool empties. Mention how the rate itself changes in your answer. **1**

……

……

……

……

……

……

……

……

……

……

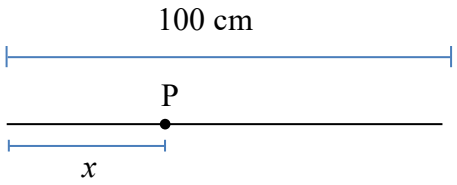
……

……

End of Question 35

Question 36 (6 marks) **Marks**

A 100 centimetre length of wire is cut into two pieces at point P, as shown in the diagram, where x is the length of one of the two pieces of wire.



a) The piece that is length x cm is used to form a circle and the other is used to form a square. **3**

Show that the total area of the circle and square can be given by :

$$A = \frac{x^2}{4\pi} + \frac{(100 - x)^2}{16}$$

……

……

……

……

……

……

……

……

……

……

……

……

……

Question 36 Continues on page 29

Question 36 Continued **Marks**

b) At what length should x be cut to minimise the total area of the circle and square? **3**
Give your answer to the nearest centimetre.

……

……

……

……

……

……

……

……

……

……

……

……

……

……

……

……

……

……

……

End of Paper

Section II extra writing space

If you use this space, clearly indicate which question you are answering.

Section II extra writing space

If you use this space, clearly indicate which question you are answering.

NAME: Solutions

CLASS: 12MTA_____ or 12MTX_____

CHERRYBROOK TECHNOLOGY HIGH SCHOOL

2020

YEAR 12

AP4

MATHEMATICS ADVANCED

Time allowed – 3 hours plus 10 minutes reading time

General Instructions

- Attempt all questions
- Write your name on the question paper
- Write using black pen
- Calculators approved by NESA may be used
- The NESA reference sheet has been provided
- For questions in Section II, show relevant mathematical reasoning and/or calculations
- Marks may not be awarded for careless, badly arranged, or poorly written work

Total marks: 100

Section I – 10 marks (pages 3 – 5)

- Attempt Questions 1-10
- Allow about 15 minutes for this section

Section II – 90 marks (pages 6 – 29)

- Attempt Questions 11-36
- Allow about 2 hours and 45 minutes for this section

BLANK PAGE

Section I

10 Marks
Attempt Questions 1 to 10
Allow about 15 minutes for this section

Use the multiple choice answer sheet for Questions 1 to 10

1 For what values of x is the curve $f(x) = 2x^3 + x^2$ concave down?

(A) $x < -\frac{1}{6}$
(B) $x > -\frac{1}{6}$
(C) $x < -6$
(D) $x > 6$

2 What is the domain of the function $f(x) = \sqrt{x} + \frac{1}{\sqrt{2-x}}$?

(A) (0, 2)
(B) [0, 2)
(C) (0,2]
(D) [0, 2]

3 Two events, A and B are independent, and $P(A)$ and $P(B)$ are both non-zero.

Which of the following expressions gives the probability that event A occurs and then event B also occurs?

(A) $P(B|A)$
(B) $P(A|B)$
(C) $P(A \cup B)$
(D) $P(A)P(B)$

4 The variance of a discrete random variable X is $\mathrm{Var}(X) = 2.6$

Calculate $\mathrm{Var}(3X - 1)$

(A) 6.8
(B) 7.8
(C) 23.4
(D) 22.4

5 What is $f'(x)$ if $f(x) = 3x^4(4-x)^3$?

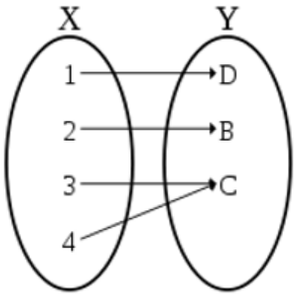
(A) $12x^3(4-x)^2(7x-16)$

(B) $12x^3(4-x)^2(16-7x)$

(C) $3x^3(4-x)^2(7x-16)$

(D) $3x^3(4-x)^2(16-7x)$

6 Which type of relation is shown below?



(A) One-to-one

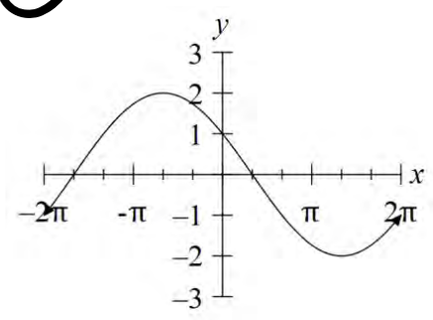
(B) One-to-many

(C) Many-to-one

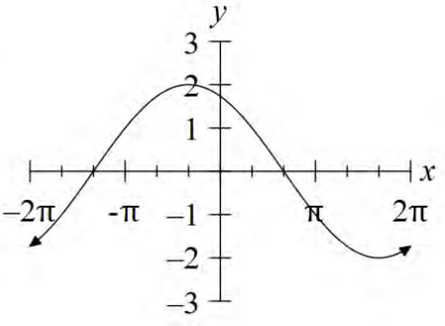
(D) Many-to-many

7 Which of the following best represents the graph of $g(x) = 2\cos\left(\frac{x}{2} + \frac{\pi}{3}\right)$?

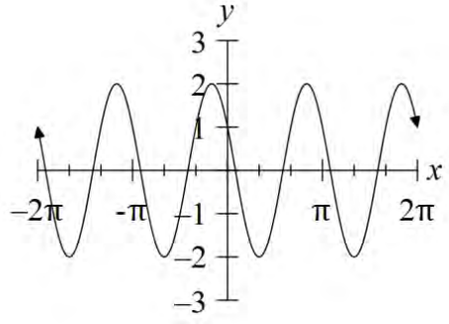
(A)



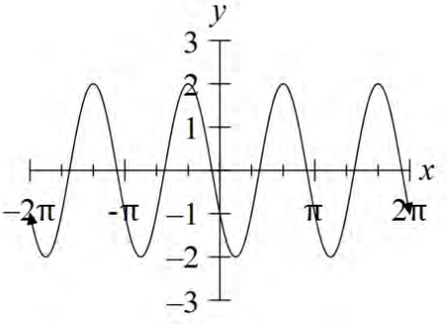
(B)



(C)



(D)



8 What is the value of $\ln 2 + \ln 4 + \ln 8 + \ln 2^{2n}$?

(A) $n^2 \ln 2$
(B) $n(n+1)\ln 2$
(C) $n(n+2)\ln 2$
(D) $n(2n+1)\ln 2$

9 What is the equation of the function $f(x)$ whose graph has undergone transformations in the following order?

- Translated left 2 units
- Horizontally dilated by a factor of 3
- Translated down 4 units

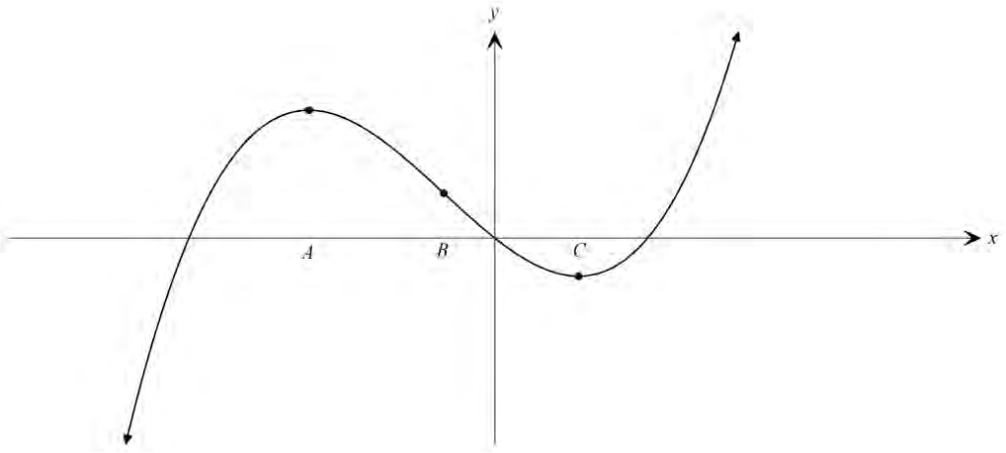
(A) $f\left(\frac{x}{3}+2\right)-4$

(B) $f\big(3(x+2)\big)-4$

(C) $f\left(\frac{x+2}{3}\right)-4$

(D) $f(3x+2)-4$

10 The graph of $y = f(x)$ is shown below.
$x = A$ and $x = C$ are stationary points, and $x = B$ is a point of inflection.



Over what domain is $f'(x) < 0$ and $f''(x) > 0$?

(A) $(-\infty, A)$
(B) (A, B)
(C) (B, C)
(D) (C, ∞)

END OF SECTION I

Section II

90 marks
Attempt Questions 11-36
Allow about 2 hours and 45 minutes for this section

Answer each question in the spaces provided.
Your responses should include relevant mathematical reasoning and/or calculations.
Extra writing space is provided at the back of the examination paper.

Marks

Question 11 (2 marks)

A geometric sequence has a first term of $\frac{1}{8}$ and the 9th term of the sequence is 8192. **2**
What is the common ratio?

$T_n = ar^{n-1}$

$8192 = \frac{1}{8} \times r^8$

$65536 = r^8$

$r = \sqrt[8]{65536}$

$r = \pm 4$

Question 12 (2 marks)

Find the exact value of **2**

$\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \cos x \, dx.$

$= [\sin x]_{\pi/4}^{\pi/3}$

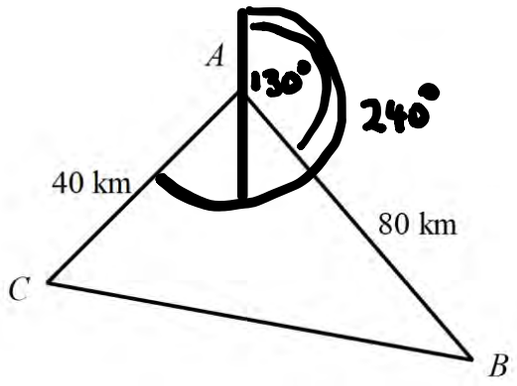
$= \sin\frac{\pi}{3} - \sin\frac{\pi}{4}$

$= \frac{\sqrt{3}}{2} - \frac{1}{\sqrt{2}}$

$= \frac{\sqrt{3} - \sqrt{2}}{2}$

Question 13 (3 marks) **Marks**

Three towns, *A*, *B* and *C* form a triangle. **3**
Town *A* is 80 km from Town *B* and Town *C* is 40 km from Town *A* as shown below:



The bearing of Town *B* from Town *A* is 130°. The bearing of Town *C* from Town *A* is 240°.

Find the distance between Town *B* and Town *C*, to the nearest kilometre.

$\angle BAC = 240° - 130° = 110°$

Using the Cosine Rule

$BC^2 = 40^2 + 80^2 - 2 \times 40 \times 80 \times \cos 110°$

$BC = \sqrt{40^2 + 80^2 - 2 \times 40 \times 80 \times \cos 110°}$

$BC = 100.94$

$= 101$ km

Question 14 (2 marks) **Marks**

Find $\frac{d}{dx}(x^3 \tan 2x)$ **2**

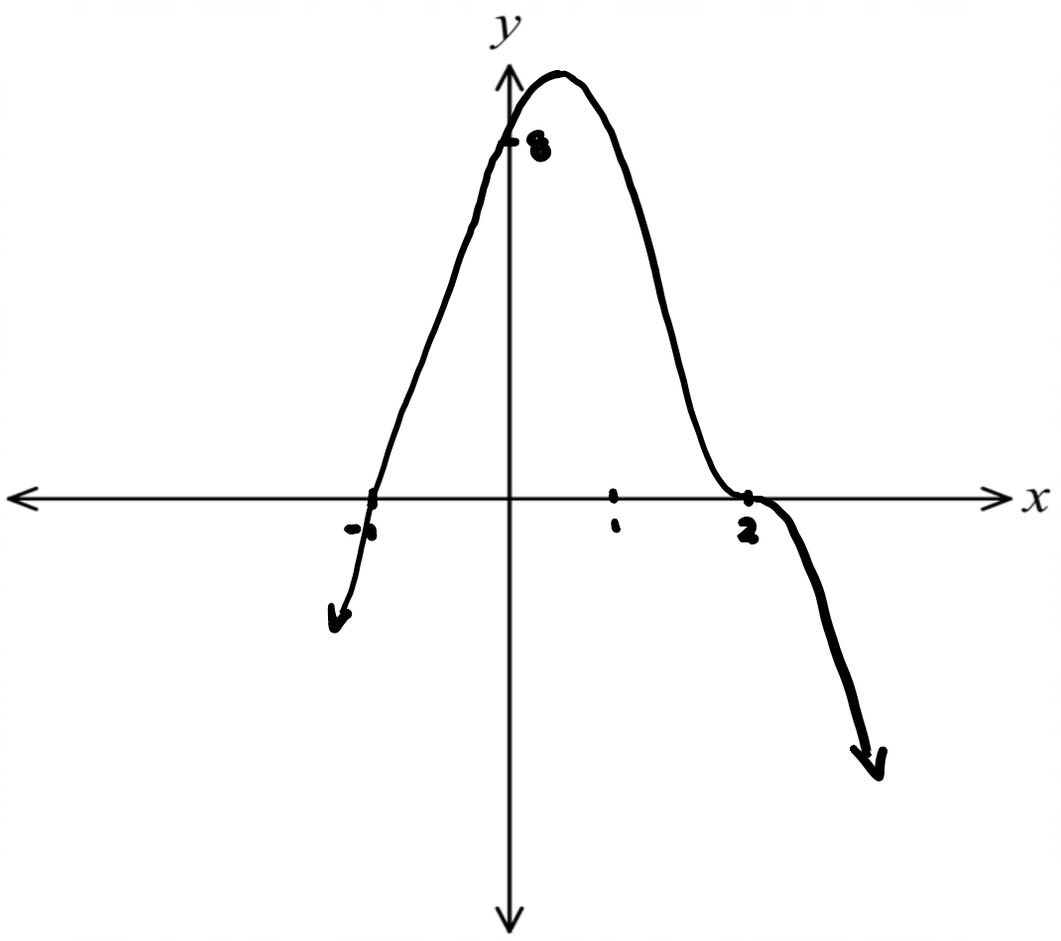
$u = x^3 \qquad v = \tan 2x$

$u' = 3x^2 \qquad v' = 2\sec^2 2x$

$$\frac{d}{dx}(x^3 \tan 2x) = 3x^2 \tan 2x + 2x^3 \sec^2 2x$$
$$= x^2(3\tan 2x + 2x \sec^2 2x)$$

Question 15 (2 marks)

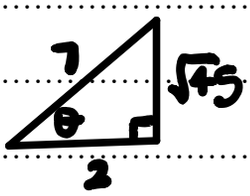
Sketch the graph of $y = -(x+1)(x-2)^3$, showing the intercepts. **2**



Question 16 (2 marks) **Marks**

Given $\cos\theta = \frac{2}{7}$ and $270° \leq \theta \leq 360°$, find the exact value of $\sin\theta$. **2**

Sin is negative in 3rd Quad.



$h = \sqrt{7^2 - 2^2}$

$= \sqrt{45}$

$\sin\theta = -\frac{\sqrt{45}}{7}$

$\sin\theta = -\frac{3\sqrt{5}}{7}$

Question 17 (2 marks)

Show that $(\tan x + \sec x)^2 = \frac{1+\sin x}{1-\sin x}$ **2**

$LHS = (\tan x + \sec x)^2$

$= \left(\frac{\sin x}{\cos x} + \frac{1}{\cos x}\right)^2$

$= \frac{(\sin x + 1)^2}{\cos^2 x}$

$= \frac{(\sin x + 1)^2}{1 - \sin^2 x}$

$= \frac{(\sin x + 1)^2}{(1 - \sin x)(1 + \sin x)}$

$= \frac{1 + \sin x}{1 - \sin x}$

$= RHS$

Question 18 (2 marks)

Solve $2\cos^2\alpha - 3\cos\alpha - 2 = 0$ for $0 \leq \alpha \leq \pi$ **2**

Let $u = \cos\alpha$

$\therefore 2u^2 - 3u - 2 = 0$

$(2u + 1)(u - 2) = 0$

$\therefore u = -\frac{1}{2}$ or 2

$\cos\alpha = -\frac{1}{2}$ or $\cos\alpha = 2$

no solution

Acute angle $= \frac{\pi}{3}$

Cos negative in 2nd Quad

$\therefore \alpha = \frac{2\pi}{3}$

Question 19 (2 marks)

Consider the functions $f(x) = e^x$ and $g(x) = \ln(x - 2)$

a) Find the composite function $f(g(x))$. **1**

$f(g(x)) = e^{\ln(x-2)}$

$= x - 2$

b) Find in interval notation the range of the composite function. **1**

$x > 2$

$\therefore$ Range is $(0, +\infty)$

Question 20 (4 marks) **Marks**

A curve with the equation $y = f(x)$, has $\frac{dy}{dx} = x^3 + 2x - 7$.

a) The point *P* (2, 4) lies on the curve. Find the equation of *y*. **2**

$y = \int (x^3 + 2x - 7)\, dx$

$y = \frac{x^4}{4} + x^2 - 7x + C$

Sub in (2, 4) $\quad 4 = \frac{16}{4} + 4 - 14 + C$

$C = 10$

$\therefore y = \frac{x^4}{4} + x^2 - 7x + 10$

b) Find the equation of the normal to the curve at point *P*, giving your answer in general form. **2**

Grad of tangent : Sub in $x = 2$ into $\frac{dy}{dx}$

Grad of tangent $= 2^3 + 2(2) - 7 = 5$

$\therefore$ Grad of Normal $= -\frac{1}{5}$

Equation of Normal

$y - y_1 = m(x - x_1)$

$y - 4 = -\frac{1}{5}(x - 2)$

$5y - 20 = -x + 2$

$x + 5y - 22 = 0$

Question 21 (7 marks) **Marks**

A function is given by $y = 2x^3 + 3x^2 - 12x - 5$

a) Find the stationary points, determine their nature, and find any points of inflection. **5**

$y' = 6x^2 + 6x - 12$

$y'' = 12x + 6$

Stat. pts occur when $y' = 0$

i.e. $6x^2 + 6x - 12 = 0$

$x^2 + x - 2 = 0$

$(x+2)(x-1) = 0$

$\therefore x = -2$ or $x = 1$

- when $x = -2$ $y = 15$ $y'' = -18$ $(y'' < 0)$

  $\therefore$ Relative maximum turning point at $(-2, 15)$

- when $x = 1$ $y = -12$ $y'' = 18$ $(y'' > 0)$

  $\therefore$ Relative minimum turning point at $(1, -12)$

- Possible Point of Inflection $y'' = 0$

i.e $12x + 6 = 0$

$12x = -6$

$x = -\frac{1}{2}$ when $x = -\frac{1}{2}$ $y = 1\frac{1}{2}$

check change in concavity

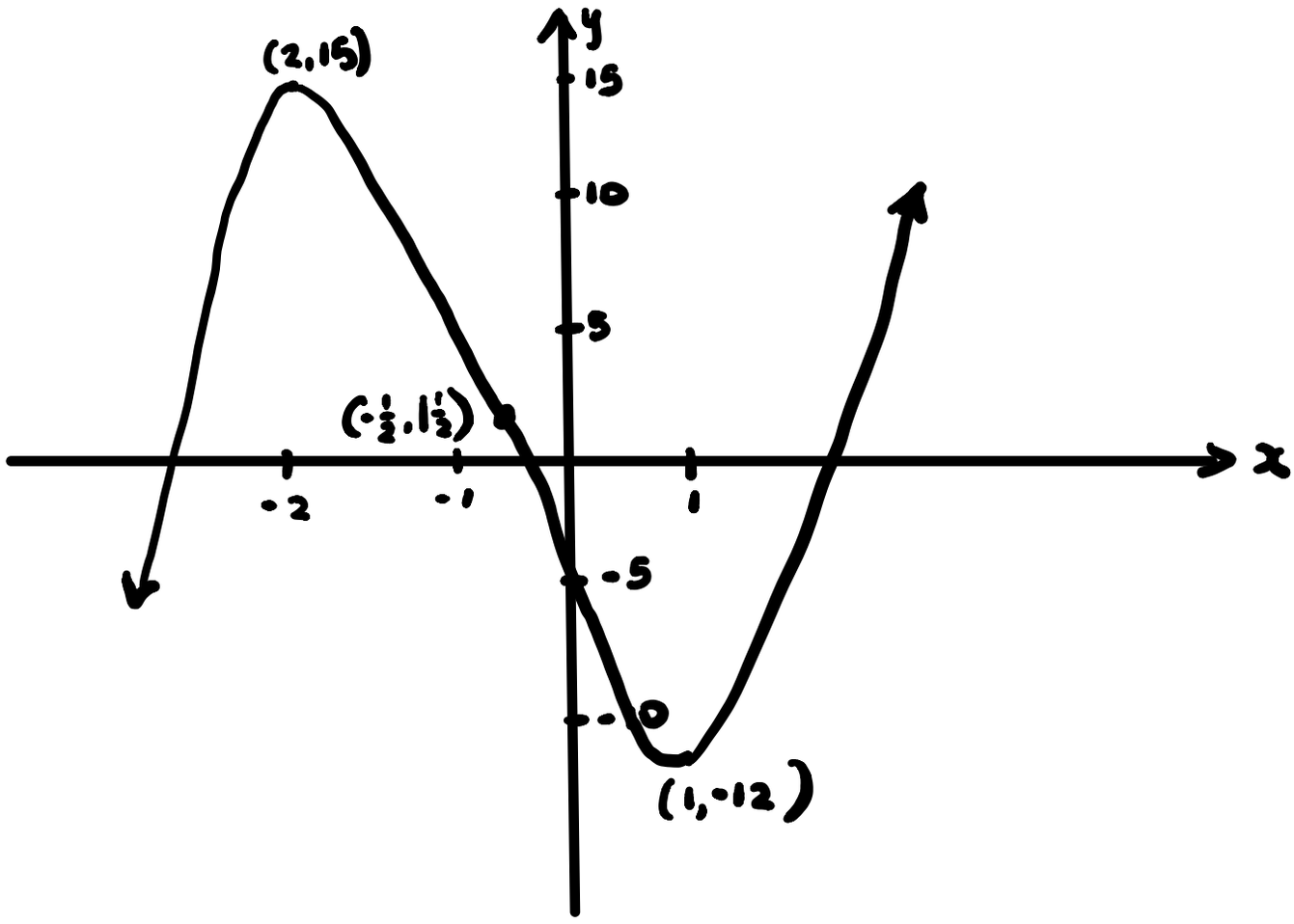
x	-1	$-\frac{1}{2}$	0
y''	-6	0	6

Since there is a change in concavity $(-\frac{1}{2}, 1\frac{1}{2})$ is a point of inflection.

Question 21 continues on page 13

Question 21 Continued **Marks**

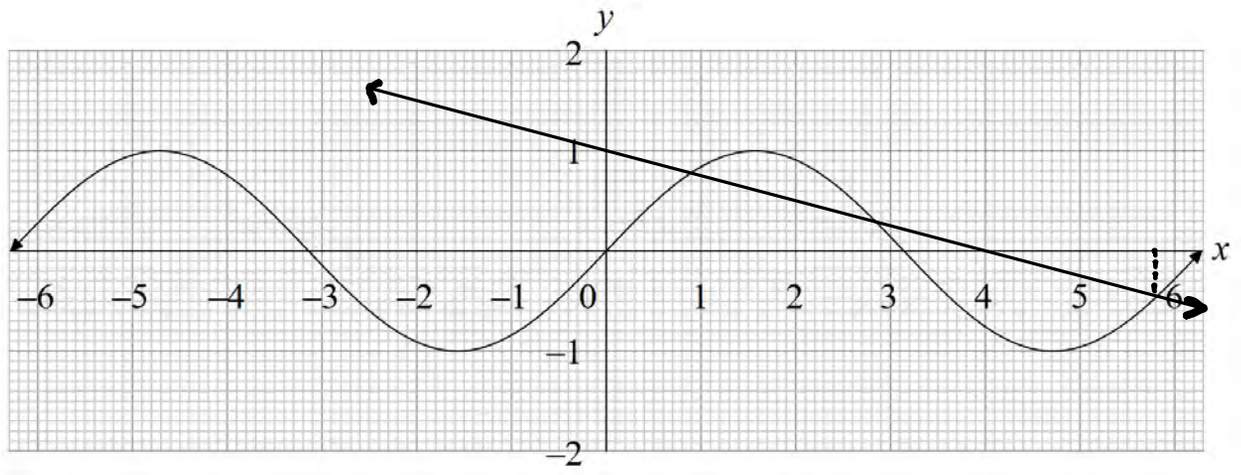
b) Sketch the curve, showing the stationary points, the point(s) of inflection and the *y*-intercept. **2**



End of Question 21

Question 22 (2 marks) **Marks**

The graph of $y = \sin x$ is shown in the diagram.



a) Explain why the equation $\sin x = 1 - \frac{x}{4}$ has three solutions. **1**

The line $y = 1 - \frac{x}{4}$ intersect $y = \sin x$

3 times

b) Use the graph to approximate the largest solution to the equation $\sin x = 1 - \frac{x}{4}$. **1**

5.8

Question 23 (2 marks)

2

If $y = \frac{e^{3x}}{x+1}$ find $\frac{dy}{dx}$

$u = e^{3x}$ $\quad v = x+1$

$u' = 3e^{3x}$ $\quad v' = 1$

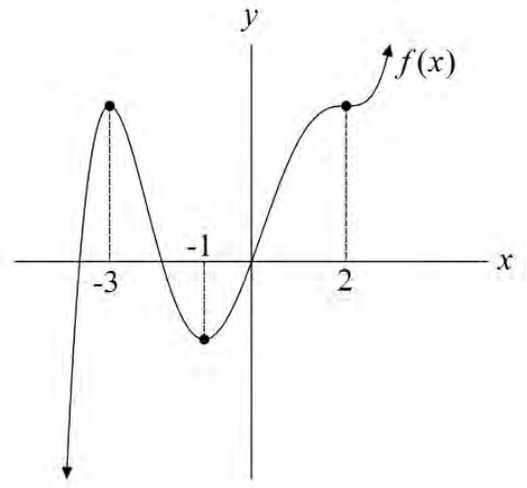
$$\frac{dy}{dx} = \frac{(x+1)3e^{3x} - e^{3x}}{(x+1)^2}$$

$$= \frac{e^{3x}(3x+3-1)}{(x+1)^2}$$

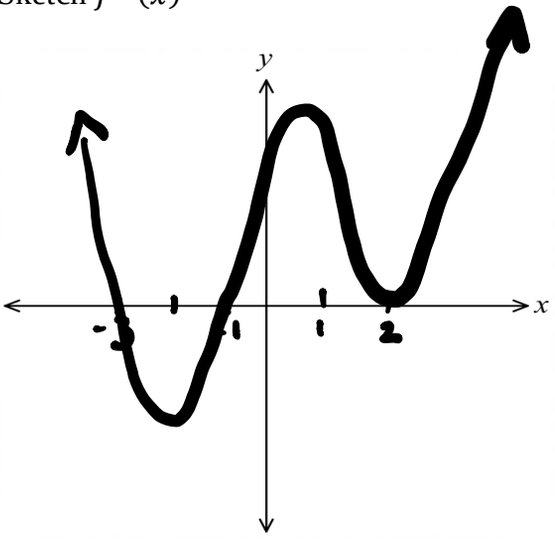
$$= \frac{e^{3x}(3x+2)}{(x+1)^2}$$

Question 24 (4 marks) **Marks**

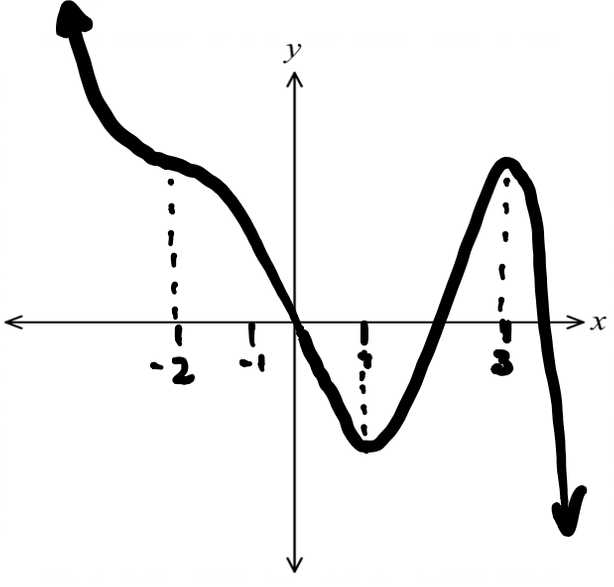
The diagram below shows the graph of $\boldsymbol{y = f(x)}$.



a) Sketch $f'(x)$ **2**



b) Sketch $f(-x)$ **2**



Question 25 (5 marks) **Marks**

a) Show that the derivative of $\ln\left(\frac{3+x}{3-x}\right)$ is $\frac{6}{9-x^2}$ **3**

$\ln\left(\frac{3+x}{3-x}\right) = \ln(3+x) - \ln(3-x)$

$\therefore \frac{d}{dx}\left(\frac{3+x}{3-x}\right) = \frac{1}{3+x} - \frac{-1}{3-x}$

$= \frac{1}{3+x} + \frac{1}{3-x}$

$= \frac{3-x+3+x}{(3+x)(3-x)}$

$= \frac{6}{9-x^2}$

b) Hence or otherwise find $\int \frac{1}{9-x^2}\,dx$. **2**

$\int \frac{1}{9-x^2}\,dx = \frac{1}{6}\int \frac{6}{9-x^2}\,dx$

$= \frac{1}{6}\ln\left|\frac{3+x}{3-x}\right| + C$

Questions 11- 25 are worth 43 marks in total.

Question 26 (3 marks) **Marks**

Use the Trapezoidal rule, with four sub-intervals, to estimate the value of $\int_0^2 \sqrt{4-x^2}\,dx$, correct to 3 decimal places. **3**

x	0	$\frac{1}{2}$	1	1.5	2
y	2	$\frac{\sqrt{15}}{2}$	$\sqrt{3}$	$\frac{\sqrt{7}}{2}$	0

$$\int_0^2 \sqrt{4-x^2}\,dx$$

$$\approx \frac{\frac{1}{2}}{2}\left[(2+0)+2\left(\frac{\sqrt{15}}{2}+\sqrt{3}+\frac{\sqrt{7}}{2}\right)\right]$$

$$\approx 2.996$$

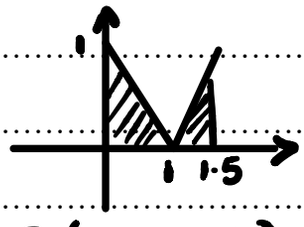
Question 27 (2 marks)

The probability density function for the continuous random variable X is given by:

$$f(x)=\begin{cases}|1-x| & 0\le x\le 2\\ 0 & \text{otherwise}\end{cases}$$

$$|1-x| = \begin{cases}1-x & x\le 1\\ -1+x & x>1\end{cases}$$

Find $P(X\le 1.5)$ **2**



$P(x\le 1.5)$

$=\left(\frac{1}{2}\times 1\times 1\right)+\left(\frac{1}{2}\times\frac{1}{2}\times\frac{1}{2}\right)$

$=\frac{1}{2}+\frac{1}{8}$

$=\frac{5}{8}$

OR

$$\int_0^{1.5}|1-x|\,dx$$

$$=\int_0^1 1-x\,dx+\int_1^{1.5}-1+x\,dx$$

$$=\left[x-\frac{x^2}{2}\right]_0^1+\left[-x+\frac{x^2}{2}\right]_1^{1.5}$$

$$=\left(1-\frac{1}{2}\right)+\left(-1.5+\frac{1.5^2}{2}\right)-\left(-1+\frac{1}{2}\right)$$

$=0.625$ or $\frac{5}{8}$

Question 28 (3 marks) **Marks**

A soft drink company produces a bottled drink. The amount of liquid poured into each bottle by the bottling machine is normally distributed, with a mean of 300 mL and a standard deviation of 10 mL.

a) A bottle has a z-score of -1.2. How many mL below the mean is this bottle? **1**

$-1.2 \times 10 = 12$ mL below the mean

$-1.2 \times 10 = 12$ mL below the mean

b) In order to comply with local regulations, at least 97.5% of bottles must contain more liquid than indicated on the label. **2**

What is the largest amount of millilitres the bottle's label should show in order to comply with the regulations?

95% of scores lie between -2 and 2 std dev.

5% of scores lie outside -2 and 2 std dev

$\therefore$ 2.5% of scores lie below -2 std deviations

-2 std dev. $= 300 - 2 \times 10 = 280$

$\therefore$ Label at most 280 mL to comply with regulation.

Question 29 (5 marks) **Marks**

It is known at the beginning of winter in a large population, 15% of the people in the population will be infected with a particular virus.

a) Two people are selected at random, find the probability that both of them have the virus. **1**

$(0.15)^2 = 0.0225$

b) Four people are selected at random, find the probability that at least one of them has the virus. Give your answer to 3 decimal places. **2**

$P(\text{at least 1 virus in a group of 4}) = 1 - P(\text{no virus in a group of 4})$

$= 1 - (0.85)^4$

$= 0.47799375$

$= 0.478$

c) What is the smallest number of people a drug company would need to test to have a greater than 95% chance that at least one of the tested people had the virus? **2**

$1 - (0.85)^x > 0.95$

$-(0.85)^x > -0.05$

$0.85^x < 0.05$

$\ln 0.85^x < \ln 0.05$

$x \ln 0.85 < \ln 0.05$

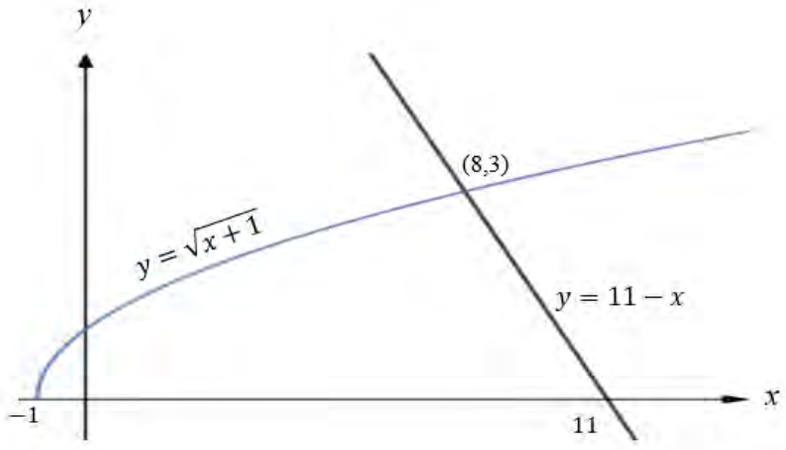
$n > \dfrac{\ln 0.05}{\ln 0.85}$

$n > 18.4$

$\therefore$ 19 people would be the smallest number to be tested to find at least 1 person with the virus.

Question 30 (3 marks) **Marks**

The functions $y = \sqrt{x+1}$ and $y = 11 - x$ are sketched below.



Calculate the area bounded by the curves $y = \sqrt{x+1}$ and $y = 11 - x$ and the x-axis. **3**

$A = \int_{-1}^{8} \sqrt{x+1}\,dx + \int_{8}^{11} 11 - x\,dx$

$= \int_{-1}^{8} (x+1)^{\frac{1}{2}}\,dx + \left[11x - \frac{x^2}{2}\right]_{8}^{11}$

$= \left[\frac{2(x+1)^{\frac{3}{2}}}{3}\right]_{-1}^{8} + \left[\left(11 \times 11 - \frac{11^2}{2}\right) - \left(11 \times 8 - \frac{8^2}{2}\right)\right]$

$= \frac{2}{3}\left[(9^{\frac{3}{2}} - 0)\right] + \frac{9}{2}$

$= 18 + \frac{9}{2}$

$= 22.5 \text{ unit}^2$

Question 31 (2 marks) **Marks**

Find the value(s) of m given that m, $3m$ and m^2+20 are consecutive terms of a geometric sequence. **2**

$$\frac{3m}{m} = \frac{m^2+20}{3m}$$

$$3 = \frac{m^2+20}{3m}$$

$$9m = m^2+20$$

$$m^2-9m+20=0$$

$$(m-4)(m-5)=0$$

$$m=4 \text{ or } m=5$$

Question 32 (5 marks)

A probability density function is defined as $f(x)=ke^{-x}$ in the domain $[0,3]$.

a) Show that $k=1.0524$, correct to four decimal places. **2**

$$\int_0^3 ke^{-x}\,dx = 1$$

$$\left[-ke^{-x}\right]_0^3 = 1$$

$$-ke^{-3}+k=1$$

$$-k(e^{-3}-1)=1$$

$$k=\frac{-1}{e^{-3}-1}$$

$$=1.0524$$

Question 32 continues on page 22

b) Hence, find the median value of the probability density function, correct to three decimal places. **3**

$$\int_0^x 1.0524e^{-x}\,dx = 0.5$$

$$\left[-1.0524e^{-x}\right]_0^x = 0.5$$

$$-1.0524e^{-x} + 1.0524 = 0.5$$

$$-1.0524e^{-x} = -0.5524$$

$$1.0524e^{-x} = 0.5524$$

$$e^{-x} = \frac{0.5524}{1.0524}$$

$$\ln e^{-x} = \ln\left(\frac{0.5524}{1.0524}\right)$$

$$-x = -0.644556$$

$$x = 0.645$$

End of Question 32

Question 33 (7 marks) **Marks**

A six-sided die is biased as shown in the probability distribution below.

x	1	2	3	4	5	6
$P(X = x)$	0.1	0.25	0.05	a	0.17	0.13

a) Explain why $a = 0.3$. **1**

$\sum P(x=x) = 1$

i.e. $0.1 + 0.25 + 0.05 + a + 0.17 + 0.13 = 1$

$\therefore \quad a = 0.3$

b) Find $P(2 < X \leq 4)$ **1**

$0.05 + 0.3 = 0.35$

c) Find $P(X \leq 4 | X > 2)$ **1**

$$\frac{0.05 + 0.3}{0.05 + 0.3 + 0.17 + 0.13} = \frac{7}{13}$$

d) Find the expected value of X. **1**

$E(x) = 1 \times 0.1 + 2 \times 0.25 + 3 \times 0.05 + 4 \times 0.3 + 5 \times 0.17 + 6 \times 0.13$

$= 3.58$

Question 33 continues on page 24

Question 33 Continued **Marks**

e) Find $E(4X + 1)$ **1**

$E(4X+1) = 4 \times 3.58 + 1$

$= 15.32$

f) Find the variance of X. **2**

$VAR(X) = (1-3.58)^2 0.1 + (2-3.58)^2 0.25$
$+ (3-3.58)^2 0.05 + (4-3.58)^2 0.3$
$+ (5-3.58)^2 0.17 + (6-3.58)^2 0.13$
$= 2.4636$

OR

$VAR(X) = (1^2 \times 0.1) + (2^2 \times 0.25) + (3^2 \times 0.05)$
$+ (4^2 \times 0.3) + (5^2 \times 0.17) + (6^2 \times 0.13)$
$- 3.58^2$

$= 2.4636$

End of Question 33

Question 34 (5 marks) **Marks**

Max conducted a survey of a group of people he knew about their age and how much they earn each week. The results are shown in the table below.

Age (years) (x)	18	45	28	15	32	68
Wage (\$/week) (W)	715	2350	1530	438	1690	1320

a) Using your calculator, find (r) Pearson's correlation coefficient correct to 2 decimal places and describe the type and strength of correlation this data gives. **2**

$r = 0.53$

moderate positive correlation

b) Using your calculator, find the equation of the least-squares regression line in the form **1**

$W = Bx + A$ where A and B are integers.

$A = 706 \quad B = 18$

$W = 18x + 706$

c) Use your equation to estimate the earnings of a 50 year-old worker. **1**

$W = 18 \times 50 + 706$

$= \$1606$

d) Your equation from part (b) cannot be used to make valid estimates for ages greater than 68 and less than 15 years. **1**

Justify this statement with calculations and/or reasons.

Not valid to extrapolate for younger & older workers - only moderate correlation coefficient

People < 15 too young to work. People > 68 retired

According to equation

8 yr old would get \$850

90 yr old would get \$2326

Question 35 (6 marks) **Marks**

A swimming pool is to be emptied for maintenance. The quantity of water, Q in litres, remaining in the pool at a time, t minutes, is given by:

$$Q(t) = 2000(25 - t)^2, \quad 0 \leq t \leq 25$$

a) At what rate (in litres/min) is the water being removed at any time (t)? **1**

$Q'(t) = -4000(25-t)$

$\therefore$ Emptying at rate of $4000(25-t)$ L/min

b) How long will it take to remove at least half of the water from the pool? **2**
Answer to the nearest minute.

Pool Full at $t=0$

$Q(0) = 2000(25-0)^2 = 1250000$ litres

half full $= 625\,000$ litres

$625000 = 2000(25-t)^2$

$312.5 = (25-t)^2$

$25 - t = \pm\sqrt{312.5}$

$t = 25 \pm \sqrt{312.5}$

$t = 7.322$ min or 42.68 min

but $0 \leq t \leq 25$

$\therefore$ It will take 8 mins to remove at least half of the water.

Question 35 Continues on page 27

Question 35 Continued **Marks**

c) At what time does the rate of flow of water from the pool reach 20 kL/minute? **2**

$20\ kL/min = 20000\ L/min$

$-20000 = -4000(25-t)$

$5 = 25-t$

$t = 20\ min$

$\therefore$ The flow rate will be 20 kL/min after 20 mins.

d) Describe how the amount of water remaining in the pool changes as the pool empties. Mention how the rate itself changes in your answer. **1**

Pool is empty when $Q(t)=0$

i.e $2000(25-t)^2 = 0$

$25-t = 0$

$t = 25\ min$

Rate: $Q'(t) = -4000(25-t)$

when $t=0$ Rate $= -100\,000$ L/min

when $t=10$ Rate $= -60\,000$ L/min

when $t=25$ Rate $= 0$ L/min

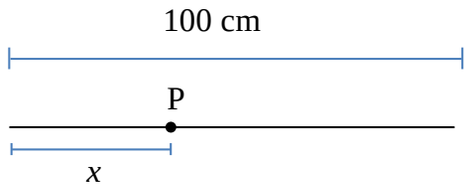
As the pool empties, the rate of flow remains negative but its value is decreasing.

i.e. The rate of flow of water is decreasing at a decreasing rate

End of Question 35

Question 36 (6 marks) **Marks**

A 100 centimetre length of wire is cut into two pieces at point P, as shown in the diagram, where x is the length of one of the two pieces of wire.



a) The piece that is length x cm is used to form a circle and the other is used to form a square. **3**

Show that the total area of the circle and square can be given by :

$$A = \frac{x^2}{4\pi} + \frac{(100-x)^2}{16}$$

Square : Perimeter = $100 - x$

1 side = $\frac{100-x}{4}$

Area = $\left(\frac{100-x}{4}\right)^2$

Circle : Circumference = x

$2\pi r = x$

$r = \frac{x}{2\pi}$

Area = $\pi\left(\frac{x}{2\pi}\right)^2 = \pi\frac{x^2}{4\pi^2}$

$= \frac{x^2}{4\pi}$

$\therefore$ Total Area = $\frac{x^2}{4\pi} + \left(\frac{100-x}{4}\right)^2$

Question 36 Continues on page 29

b) At what length should x be cut to minimise the total area of the circle and square? Give your answer to the nearest centimetre. **3**

$$A = \frac{x^2}{4\pi} + \left(\frac{100-x}{16}\right)^2$$

$$A' = \frac{2x}{4\pi} - 2\left(\frac{100-x}{16}\right)$$

$$= \frac{x}{2\pi} - \frac{100-x}{8}$$

Stat pts occur when $A' = 0$

i.e. $\frac{x}{2\pi} - \frac{100-x}{8} = 0$

$$\frac{4x - \pi(100-x)}{8\pi} = 0$$

$$4x - 100\pi + \pi x = 0$$

$$x(4+\pi) = 100\pi$$

$$x = \frac{100\pi}{4+\pi}$$

$$= 43.99$$

$$= 44\text{cm}$$

check it is a min t.p.

$$A'' = \frac{1}{2\pi} + \frac{1}{8}$$

$$= 0.284$$

> 0 for all x

$\therefore$ 44 cm is a minimum

End of Paper

$\therefore$ wire should be cut at 44cm to minimise the total area of the square and circle